A BRIEF OVERVIEW OF STOCHASTIC ANALYSIS AND SOME OF ITS APPLICATIONS

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NEWTONIAN DYNAMICS

Let's start our story in 1687, the publication year of Sir Isaac NEWTON's "PHILOSOPHIAE NATURALIS PRINCIPIA MATHEMATICA".

Here, we have the realization of a 19-century old dream by another great mind: the Archimedean dream

« Δός μοι ποῖ στῶ, καί τάν γᾶν κινήσω. »

"Give me a place to stand, and I shall move the earth."

Newton realizes it as:

"Give me your initial condition, and your dynamics, and I'll give you your position at any later time."

More precisely, given a "velocity vector field"

$$[0,\infty) \times \mathbb{R}^n \ni (t,x) \longrightarrow b(t,x) \in \mathbb{R}^n$$

and a starting position $\xi \in \mathbb{R}^n$, solve the EQUATION OF MOTION in its *integral form*

$$X(t) = \xi + \int_0^t b(s, X(s))\, ds \,, \qquad 0 \le t < \infty$$

or in its equivalent *differential form*

$$\dot{X}(t) = b(t, X(t)) \,, \qquad X(0) = \xi \,.$$

Notions like "derivative" and "integral", or results like the "fundamental theorem of calulus" and the CHAIN RULE

$$f(X(t)) = f(\xi) + \int_0^t f'(X(s))\, dX(s) \,, \quad 0 \le t < \infty$$

that connect one to the other, we take today for granted. NEWTON had to develop them out of thin air.

O.K., how do you solve an equation like

$$X(t) = \xi + \int_0^t b(s, X(s))\, ds, \quad 0 \le t < \infty ?$$

Well, knowing nothing better to do, you try SUCCESSIVE APPROXIMATIONS

$$X^{(0)}(t) \equiv \xi, \qquad 0 \le t < \infty$$

$$X^{(1)}(t) \equiv \xi + \int_0^t b(s, \xi)\, ds, \qquad 0 \le t < \infty$$

...

$$X^{(n+1)}(t) \equiv \xi + \int_0^t b(s, X^{(n)}(s))\, ds, \quad 0 \le t < \infty$$

....

with PICARD-LINDELÖF; then pray that the limit

$$X(t) \triangleq \lim_{n \to \infty} X^{(n)}(t), \qquad 0 \le t < \infty$$

exists in $\mathbb{R}^n$ (i.e., the thing does not explode) and solves the equation we started with.

It's not very hard to show that all this can be done under the following conditions:

LIPSCHITZ : $|b(t,x) - b(t,y)| \leq K\,|x-y|$

LINEAR GROWTH: $|b(t,x)| \leq K\,(1+|x|)$

for some real constant $K>0$, and all $t \in [0,\infty)$, $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$.

BROWNIAN MOTION

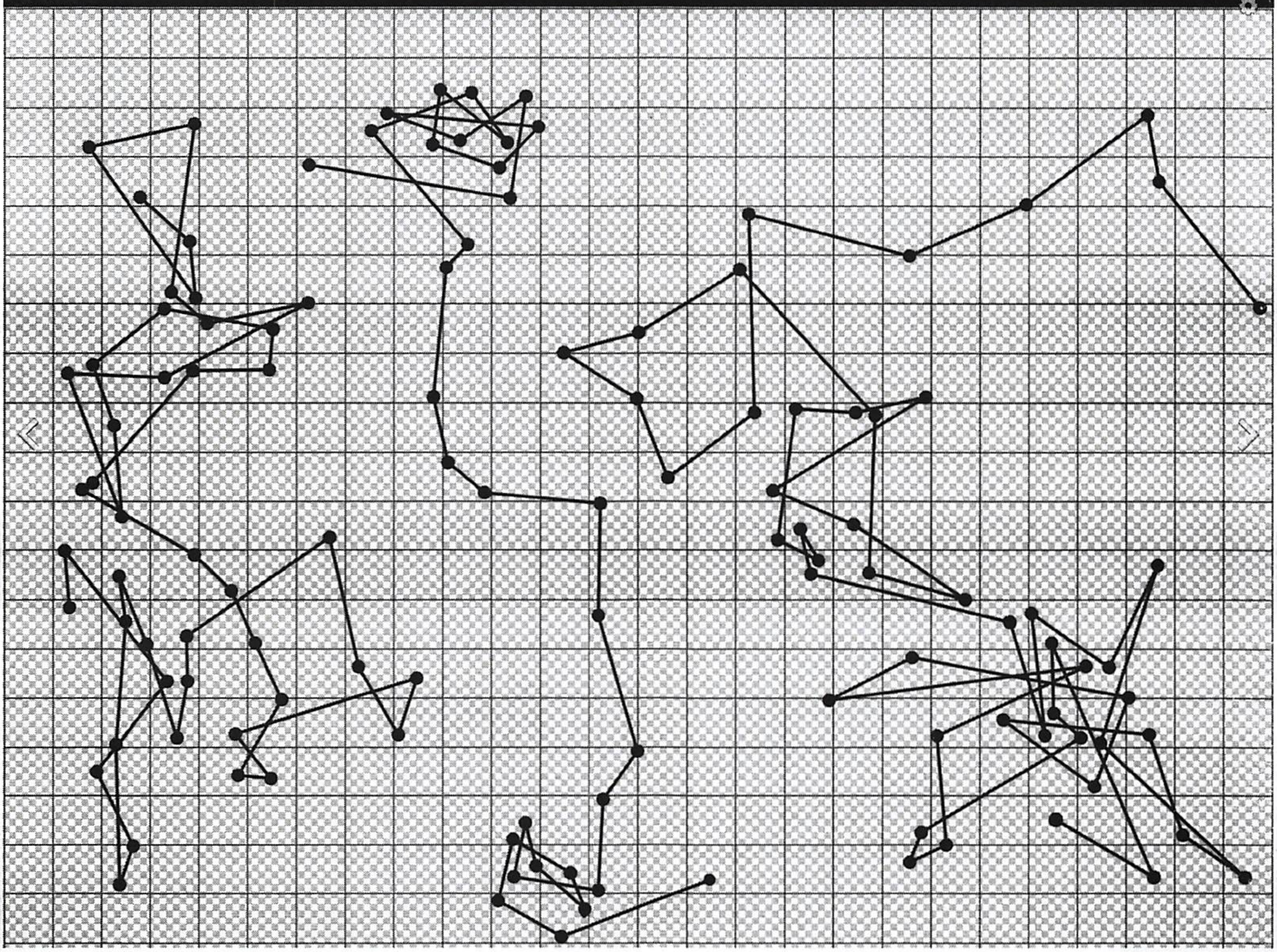
Now let's go from Cambridge in 1687 to Edinburgh in 1828, and to NEWTON's Scottish compatriot, the botanist Robert BROWN.

This guy put grains of pollen in water, and observed for the first time the particulate and very irregular motion posited by the pre-Socratic philosophers Δημόκριτος and Ἐπίκουρος, and expounded upon by the Roman philosopher/poet LUCRETIUS in his "De Rerum Natura".

We call this BROWNIAN MOTION today. A physical theory for it had to wait until 1905, the "annus mirabilis" of Albert EINSTEIN. This theory put to rest all remaining doubts about the molecular/atomic nature of matter.

It paved also the way for the measurement of the AVOGADRO number, via experimentation.

5.a



Reproduced from the book of Jean Baptiste Perrin, *Les Atomes*, three tracings of the motion of colloidal particles of radius 0.53 μm, as seen under the microscope, are displayed. Successive positions every 30 seconds are joined by straight line segments (the mesh size is 3.2 μm).[7]

... More details

J. B. Perrin, SVG drawing by MiraiWarren - SVG drawing based on File:PerrinPlot2.gif, itself from J. B. Perrin, "Mouvement brownien et réalité moléculaire," *Ann. de Chimie et de Physique* (VIII) 18, 5-114 (1909).

File: PerrinPlot2.svg

Created: 1 January 1909

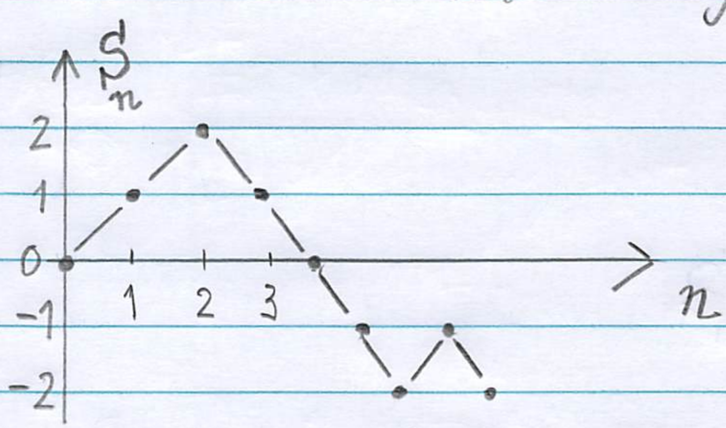
Reproduced from Jean Baptiste Perrin, "Mouvement brownien et réalité moléculaire," Ann. de Chimie et de Physique (VIII) 18, 5-114, 1909 (a slightly different version appeared in the book Les Atomes). Three tracings of the motion of colloidal particles of radius 0.53 μm, as seen under the microscope, are displayed. Successive positions every 30 seconds are joined by straight line segments (the mesh size is 3.2 μm)

J. B. PERRIN: Mouvement BROWNien et Réalité Moleculaire

RANDOM WALK

Here is a one-dimensional caricature of this motion. At any time-point $n = 1, 2, 3, \ldots$ a "particle" (drunk, our monetary fortune, ...) is kicked either up, or down, by a unit of space, each with probability $1/2$. We denote these independent kicks by $\eta_1, \eta_2, \ldots$ and the position of the particle at time n by

$$S_n = \sum_{j=1}^{n} \eta_j .$$



This is the simple, symmetric random walk : $\mathbb{P}(\eta_j = \pm 1) = 1/2$.

This walk goes all over the place, visits every site on the integer lattice, but "slowly".

- Its growth is not ballistic (proportional to n) but diffusive (proportional to $\sqrt{n}$); in fact

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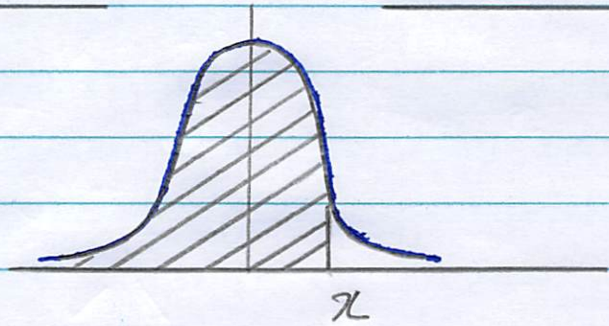
$$\frac{S_n}{\sqrt{n}} \xrightarrow[n\uparrow\infty]{\mathcal{D}} Z = \text{standard Gaussian}$$

De MOIVRE (1728)

meaning

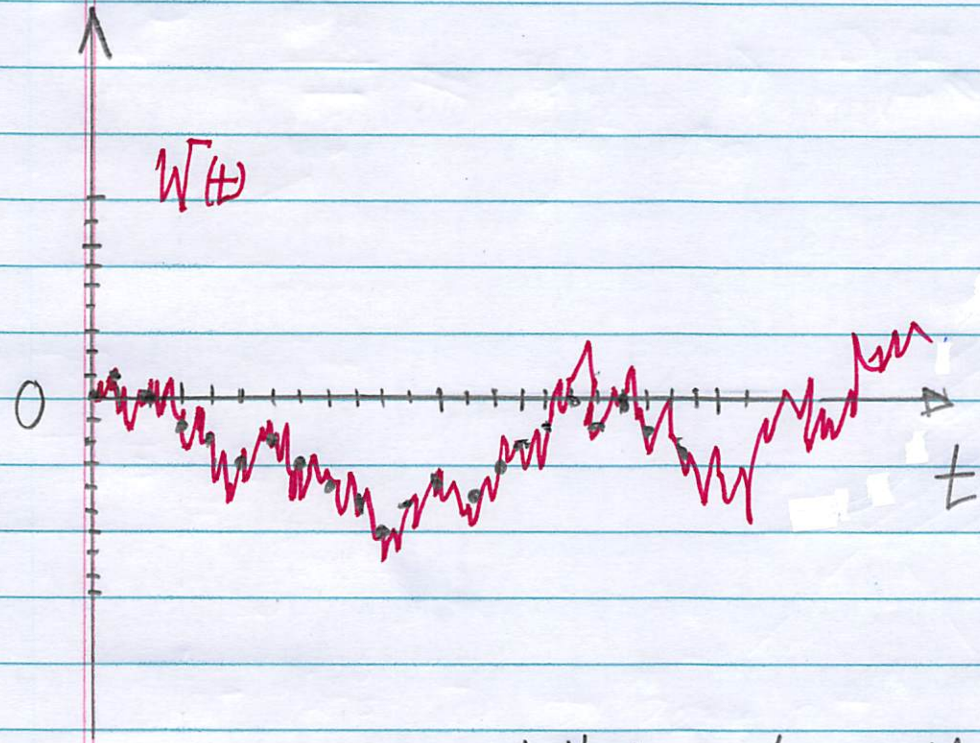
$$\mathbb{P}\left(\frac{S_n}{\sqrt{n}} \leq x\right) \xrightarrow[n\uparrow\infty]{} \mathbb{P}(Z \leq x) = \int_{-\infty}^{x} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\,du$$

CENTRAL LIMIT THEOREM



Now let's try to "zoom out" of this picture: make the time δ between "kicks" very small (a few ÅNGSTRÖM), and the size h of the kicks also very small (a few nanoseconds), and keep sending them both to zero. But in a clever way, so the thing does not explode in our face, and we "don't throw the baby out with the bathwater."

We get now the "zoomed out" random walk, as a function of time $t \in [0, \infty)$:



$$h \sum_{j=1}^{[t/\delta]} \eta_j$$

Independent $\eta_1, \eta_2, \dots$ with $P(\eta_j = \pm 1) = \frac{1}{2}$.

What is here the correct scaling?
Well, the one suggested by the DE MOIVRE
C.L. Theorem:

$$\delta = 1/n \;, \quad h = 1/\sqrt{n}.$$

In fact

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{\lfloor nt \rfloor} \eta_j \xrightarrow{\mathcal{D}} W(t):$$ Gaussian, with mean zero and variance t.

$$(*) \quad \mathbb{P}\big(W(t) \in A\big) = \int_A p(t,x)\,dx\,, \qquad p(t,x) \triangleq \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}$$

Heat Equation: $\dfrac{\partial p}{\partial t} = \dfrac{1}{2}\,\dfrac{\partial^2 p}{\partial x^2}$

We get in the limit a continuous picture, as we zoom out. The family of random variables $\big(W(t)\big)_{0 \le t < \infty}$ has the following properties, posited by EINSTEIN as BROWNIAN MOTION:

(i) $W(0) = 0$,

(ii) For $0 < t_1 < t_2 < \dots < t_k < \infty$, the increments $\big(W(t_j) - W(t_{j-1})\big)_{j=1,\dots,k}$ are independent.

(iii) Every increment $W(t_j) - W(t_{j-1})$ has Gaussian distribution as in $(*)$ above, with $t = t_j - t_{j-1}$.

(iv) The function $t \mapsto W(t)$ is continuous.

A mathematical proof for the existence (and construction, in terms of random trigonometric series) of such an object, was given by N. WIENER (1923).

The resulting random BROWNIAN Motion is continuous, but very rough. <u>It is nowhere differentiable.</u>

. It has infinite first variation, but finite second variation:

$$\sum_{j=1}^{2^n} \left| W\left(\frac{jt}{2^n}\right) - W\left(\frac{(j-1)t}{2^n}\right) \right| \xrightarrow[n\to\infty]{} \infty$$

$$\sum_{j=1}^{2^n} \left| W\left(j\frac{t}{2^n}\right) - W\left((j-1)\frac{t}{2^n}\right) \right|^2 \xrightarrow[n\to\infty]{} t \ .$$

<u>MARKOV PROPERTY:</u> At any $t \in [0,\infty)$, the motion "starts afresh", in the sense that $\big(W(t+u) - W(t)\big)_{u\geq 0}$ is BROWNIAN Motion, and independent of the "past" $W(s)$, $0 \leq s \leq t$.

In particular, it has the

MARTINGALE PROPERTY: $\mathbb{E}\left[W(t+u) \mid W(s),\ 0 \le s \le t\right] = W(t)$.

J. L. DOOB (1953).

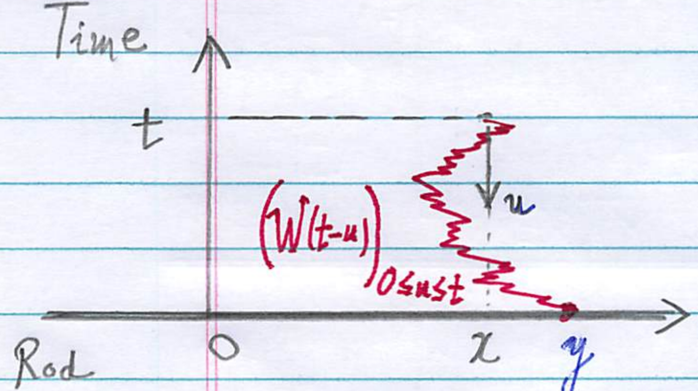
Exercise: Verify the martingale property of $\left(W^2(t) - t\right)_{t \ge 0}$,

$$\mathbb{E}\left[W^2(t+u) - (t+u) \mid W(s),\ 0 \le s \le t\right] = W^2(t) - t.$$

HEAT TRANSFER: Take an infinite iron rod, and specify temperatures $f(y)$, $y \in \mathbb{R}$ at time $t = 0$.

Suppose $f: \mathbb{R} \to [0, \infty)$ is a bounded, continuous function.

Then the temperature at time $t > 0$ and position $x \in \mathbb{R}$, is given by



$$u(t,x) = \mathbb{E}\left[f(x + W(t))\right] = \int_{\mathbb{R}} \frac{e^{-(y-x)^2/2t}}{\sqrt{2\pi t}} f(y)\, dy$$

Run a BROWNian motion backwards, take the average value of f over all possible landings.

11.2

This function $\left(u(t,x)\right)_{\substack{t>0 \\ x\in\mathbb{R}}}$ solves the <u>HEAT EQUATION</u>

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2},$$

subject to the <u>initial condition</u>

$$\lim_{t\downarrow 0} u(t,x) = f(x)$$

for every $x\in\mathbb{R}$.

<u>Martingale Property:</u> It is hard to believe, but absolutely, that almost the entire theory of Probability can be developed by mastering this innocuous-seeming property

$$\mathbb{E}\left(X(t) \mid X(u),\ 0\le u\le s \right) = X(s).$$

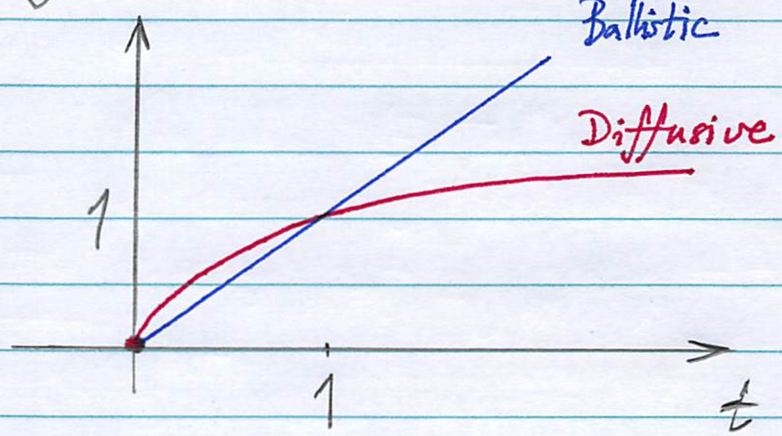
This theory "sprang fully armed from the forehead of JOE DOOB".

<u>NOTE</u>: It is wort pointing out here, the <u>diffusive displacement</u>

$$E|W(t)| = c\sqrt{t}\ , \quad t \geq 0$$

of this motion; as opposed to "ballistic" displacement $b \cdot t$, with b the prevalent velocity.

Here $b = c = 1$.



Ballistic motion (trend, velocity) gets you a lot farther than diffusive motion (noise), if you give it time.

<u>But noise dominates trend over short time scales.</u> (Not a very good idea to make investment decisions, for instance, based on daily fluctuations of assets. There give you mostly the noise.)

P. LÉVY

BROWNian Motion is an inexhaustible font of deep, fascinating mathematics. Nobody understood it better than Paul LÉVY, who studied many of its deepest properties in the 1930's and 40's.

P. LÉVY's dream: Can the NEWTONian view of mechanics be extended, to account for particulate motion not only subjected to a velocity field $b(t,x)$, but also to a local dispersion $\sigma(t,x)$ due to BROWNian motion ?

Taking $n=1$ to keep notation simple, can meaning be given to the stochastic integral equation

$$X(t) = \xi + \int_0^t b(s, X(s))\, ds + \int_0^t \sigma(s, X(s))\, dW(s)$$

for all $t \geq 0$ or, in differential form:

$$\begin{cases} dX(t) = b(t, X(t))\, dt + \sigma(t, X(t))\, dW(t) \\ X(0) = \xi. \end{cases}$$

K. ITÔ (1942)

This dream was made a reality during the war, in Nagoya, Japan, by a then completely unknown 27-year old. (Didn't take 19 centuries this time...)

Imagine trying to repeat the PICARD-LINDELÖF scheme

Manageable...
WIENER Integral

$$X^{(0)}(t) \equiv \xi$$

$$X^{(1)}(t) \equiv \xi + \int_0^t b(s,\xi)\,ds + \underline{\int_0^t \sigma(s,\xi)\,dW(s)}$$

$$\text{“}\sigma(t,\xi)W(t) - \int_0^t W(s)\,\frac{\partial}{\partial s}\sigma(s,\xi)\,ds\text{”}$$

But now

$$X^{(2)}(t) \equiv \xi + \int_0^t b(s, X^{(1)}(s))\,ds + \underline{\int_0^t \sigma(s, X^{(1)}(s))\,dW(s)}$$

¡ RED ALERT !

What does it mean, to "integrate" with respect to $W(\cdot)$ a random function, just as rough as BROWNian motion itself?

For instance: What is $\int_0^T W(t)\, dW(t)$?

Nobody at the time
(not KOLMOGOROV, not WIENER, not LÉVY, not DOOB)
knew how to do that.

ITÔ succeeded; he defined $\int_0^T H(t)\, dW(t)$
for a vast class of random functions $H(\cdot)$ (*) as a limit in probability

$$\int_0^T H(t)\, dW(t) \equiv \mathbb{P}\text{-}\lim_{n\to\infty} \sum_{j=1}^{2^n} H\left(\frac{(j-1)T}{2^n}\right)\left[W\left(\frac{jT}{2^n}\right) - W\left((j-1)\frac{T}{2^n}\right)\right].$$

BROWNIAN increment "sticks" out into the future.

This ensured that the resulting stochastic integral $\left(\int_0^T H(t)\, dW(t)\right)_{T\geq 0}$ "inherits" good martingale properties from $W(\cdot)$.

He made also ingenuous use of the fact that the quadratic variation of $W(\cdot)$ over $[0,T]$, is T (page 10).

(*) Such that $H(t)$ is a functional of $W(s)$, $0 \leq s \leq t$ for every $t \geq 0$, and $\int_0^T H^2(t)\, dt < \infty$.

Exercise: Verify "by hand" (no fancy theories...)

$$\int_0^T W(t)\,dW(t) = \frac{1}{2}\left(W^2(T) - T\right).$$

Our old friend, the martingale of page 11....

ITÔ also proved the change-of-variable formula, or "chain rule", for the new calculus he developed:

"ITÔ's RULE"

$$f(W(T)) = f(W(0)) + \int_0^T f'(W(s))\,dW(s) + \frac{1}{2}\int_0^T f''(W(s))\,ds$$

for every $f \in C^2(\mathbb{R})$. You need here the additional smoothness and therefore the extra term in the end, because you have to go to the <u>second</u> term in TAYLOR expansion, in order to find finite variation.

ITÔ also showed that, under the LIPSCHITZ and linear growth conditions on both $b(t,x)$ and $\sigma(t,x)$, the PICARD-LINDELÖF scheme started on p.13 converges to <u>the</u> solution $X(\cdot)$ of the STOCHASTIC INTEGRAL EQUATION

$$X(t) = \xi + \int_0^t b(s, X(s))\,ds + \int_0^t \sigma(s, X(s))\,dW(s), \quad 0 \le t < \infty.$$

This solution has the MARKOV property: at every fixed time T, it "forgets its past" and starts afresh at $\xi = X(T)$, going forward.

Consider now the equation

$$X(t) = X(0) + \int_0^t b(X(s))\,ds + \int_0^t \sigma(X(s))\,dW(s), \quad 0 \le t < \infty$$

---

with $X(0)$ a random variable with given distribution

$$\mu(A) = \int_A p_0(x)\,dx$$

and independent of the driving BROWNian motion $W(\cdot)$.

Then we have the infinitesimal generator computation

$$\lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}\Big[f(X(h)) - f(X(0)) \,\Big|\, X(0) = x \Big] = b(x) f'(x) + \frac{1}{2}\sigma^2(x) f''(x)$$

$$=: (\mathcal{A}f)(x).$$

Exercise: Try your hand at this computation, say for $f \in C_b^2(\mathbb{R})$.

For every $t > 0$, the random variable $X(t)$ has then probability density function

$$\mathbb{P}(X(t) \in A) = \int_A p_t(x)\, dx$$

which satisfies the FOKKER–PLANCK, or Forward KOLMOGOROV, equation

$$\frac{\partial}{\partial t} p_t(x) = \mathcal{A}^* p_t(x).$$

Here $\mathcal{A}^*$ is the formal adjoint of the second-order differential operator $\mathcal{A}$ on page 16.

<u>Exercise:</u> A standard GAUSSIAN random variable Z, with $\mathbb{P}(Z \le z) = \int_{-\infty}^{z} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx$, has $\mathbb{E}(Z) = 0$,

$$\mathbb{E}(Z^2) = 1.$$

Given this information only, can you find $\mathbb{E}(Z^4)$ without doing any computation at all? How about $\mathbb{E}(Z^{2k})$, $k \in \mathbb{N}$?

(<u>Hint</u>: Run BROWNian motion $(W(t))_{0 \le t \le 1}$, let <u>IT</u> do the work for you.)

17.a

<u>Hint:</u> Use freely the fact that, for an integrand $H(\cdot)$ as on page 14, satisfying the conditions of the footnote there, as well as the "finite energy" condition

$$\mathbb{E}\int_0^T H^2(t)\,dt < \infty, \qquad \forall\ T\in[0,\infty),$$

the <u>ITÔ integral</u> process

$$I(t) \triangleq \int_0^t H(s)\,dW(s), \quad 0\le t<\infty$$

<u>is itself a martingale,</u> in fact square-integrable: we have the so-called

[ITÔ ISOMETRY]

$$\mathbb{E}\left(\int_0^T H(t)\,dW(t)\right)^2 = \mathbb{E}\int_0^T H^2(t)\,dt,$$

in addition to

$$\mathbb{E}\int_0^T H(t)\,dW(t) = 0.$$

FILTERING

Suppose now, that it is not possible to observe directly the evolution of the diffusive motion $X(\cdot)$ as in

[SIGNAL] $$X(t) = X(0) + \int_0^t b(X(s))\,ds + \int_0^t \sigma(X(s))\,dW(s)$$

of page 16. Instead, we observe a function $h(t, X(t))$ of it, "in a bath of white noise"

[OBS.] $$Y(t) = \int_0^t h(s, X(s))\,ds + B(t), \quad t \geq 0.$$

Here, $B(\cdot)$ is BROWNIAN Motion, independent of the "signal" $X(\cdot)$.

AM : $$h(t,x) = x \cdot \cos(w_c t)$$

FM : $$h(t,x) = A_c \cdot \cos(xt).$$

Armstrong (1940)

What is now the conditional distribution of the signal $X(t)$, given the observations $Y(s),\ 0 \leq s \leq t$ accumulated up to time t ?

In particular, what can we say about the — now random — <u>conditional probability density function</u>

$$\mathbb{P}\left(X(t)\in A \,\middle|\, Y(s), 0\le s\le t\right)=\int_A p_t(x)\,dx$$

of the current signal, given the heretofore observations?

This random probability density function $p_t(\cdot)$ is now governed by the KUSHNER - STRATONOVICH Equation (1964)

$$dp_t(x)=\mathcal{A}^* p_t(x)\,dt+\left(h(t,x)-\int_{\mathbb{R}} h(t,\xi)\,p_t(\xi)\,d\xi\right)p_t(x)\,.\,dN(t)$$

A stochastic partial differential equation of 2^{nd} order. Nonlinear; non-local. Driven by the "innovations process"

$$N(t)\triangleq Y(t)-\int_0^t \hat{H}(s)\,ds\,,\qquad \hat{H}(t)\triangleq \mathbb{E}\left(h(t,X(t))\,\middle|\,Y(s),0\le s\le t\right)$$

$$=\int_{\mathbb{R}} h(t,\xi)\,p_t(\xi)\,d\xi\,,$$

a BROWNIAN Motion on its own right, with $N(t)$ a functional of $Y(s), 0\le s\le t$.

In general, this equation cannot be solved explicitly. The BIG EXCEPTION is the case

$$h(t,x) = x, \qquad X(\cdot) \text{ ORNSTEIN-UHLENBECK}$$

$$dX(t) = a\,X(t)\,dt + dW(t), \qquad p_0(x) = \frac{e^{-(x-m)^2/2v}}{\sqrt{2\pi v}}$$

$$dY(t) = X(t)\,dt + dB(t).$$

Then the conditional distribution

$$\mathbb{P}\big(X(t) \in A \,\big|\, Y(s),\, 0 \le s \le t\big) = \int_A p_t(x)\,dx$$

$$p_t(x) = \frac{e^{-\frac{(x-\hat{X}(t))^2}{2V(t)}}}{\sqrt{2\pi V(t)}}$$

is GAUSSian, with (non-random) variance $V(t)$ given by the solution of the RICCATI Equation

$$\frac{d}{dt} V(t) = 1 + 2a\,V(t) - V^2(t), \qquad V(0) = v$$

and (random) mean $\hat{X}(t) = \mathbb{E}\big[X(t) \,|\, Y(s),\, 0 \le s \le t\big]$ which satisfies the

R.E. KALMAN (1960) Filter stochastic differential Equation

$$d\hat{X}(t) = a\,\hat{X}(t)\,dt + V(t)\,dN(t)$$
$$= \big(a - V(t)\big)\hat{X}(t)\,dt + V(t)\cdot dY(t)\,, \qquad \hat{X}(0) = m.$$

One of the greatest successes of Probability and Stochastics, EVER.

It, and the theory of the Linear/Quadratic/Gaussian Regulator, also developed by Rudolf KÁLMÁN, are the tools that got us to the Moon in the 1960's.

NONLINEAR SPDE's ARE NOW UBIQUITOUS:

Ferromagnetism near criticality; Interface Growth Phenomena; Dynamics of 2D Coulomb Gases; Mathematical Biology; Fluids stirred by random forcing; Fluctuations around mean-field limits; Porous-Medium equations for ranked-based interacting particles; Forward Stochastic Utilities.

<u>Exercise</u>: Write down explicitly the solution of the KALMAN Filter equation of

$$\boxed{d\hat{X}(t) = (a - V(t))\hat{X}(t) + V(t) . dY(s)} \, ,$$

i.e., express the conditional expectation

$$\hat{X}(t) = \mathbb{E}\left(X(t) \,/\, Y(s), \, 0 \le s \le t\right),$$

the mean of the conditional distribution of $X(t)$, given the record of observations $Y(s), \, 0 \le s \le t$, as a functional of these observations.

(<u>Hint</u>: As with linear ODE's, introduce the integrating factor $\beta(t) \triangleq \exp\left(-\int_0^t (\alpha - V(s))\, ds\right)$, and apply the usual product rule to $\beta(t)\hat{X}(t)$, as well as the usual integration-by-parts. Without fear or trepidation: they are valid in the present context.)

SEQUENTIAL ANALYSIS

Let us recall the process $X(\cdot)$ of

$$X(t) = X(0) + \int_0^t b(X(s))\,ds + \int_0^t \sigma(X(s))\,dW(s), \quad t \geq 0$$

as on page 16.

A random variable τ with values in $[0,\infty]$ is called STOPPING TIME for $X(\cdot)$, if

$$\mathbb{1}_{\{\tau > t\}} \text{ is a function of } X(s),\ 0 \leq s \leq t$$

for every $t \in [0,\infty)$. We denote by $\mathcal{S}$ the collection of all such random times.

For instance, given a closed subset B of the real line,

$$\begin{aligned} \tau &= 1^{st}\ t \geq 0; \text{ s.t. } X(t) \in B \\ &= \infty; \text{ if } X(t) \notin B,\ \forall\, t \geq 0 \end{aligned}$$

is a stopping time, because $\{\tau > t\} = \{X(s) \notin B,\ \forall\, 0 \leq s \leq t\}$.

On the trivial side, every $\tau = t$ is in $\mathcal{S}$.

Consider now a "reward function" $g : \mathbb{R} \to [0, \infty)$, which is continuous. If we stop the process $X(\cdot)$ at time τ, we collect the reward $g(X(\tau)) \mathbb{1}_{\{\tau<\infty\}}$. What is the maximal expected reward

$$v(x) := \sup_{\tau \in \mathcal{S}} \mathbb{E}\left[g(X(\tau)) \mathbb{1}_{\{\tau<\infty\}} \,\Big|\, X(0) = x \right]$$

that can be achieved over stopping times, when starting in position x? Note: $v(x) \geq g(x)$. Is the supremum attained? If so, can we describe a stopping rule $\tau_* \in \mathcal{S}$ that does this job?

This problem of OPTIMAL STOPPING is at the heart of Sequential Statistical Analysis, a subject with deep roots at COLUMBIA.

A. WALD, J. WOLFOWITZ (1940's)

H. ROBBINS, D.O. SIEGMUND, Y.S. CHOW, T.L. LAI.

Also J.L. DOOB, L. SNELL, ...

The problem has a beautiful solution when $X(\cdot)$ is in fact Brownian motion $W(\cdot)$ on the unit interval $[0,1]$, with absorption upon hitting one of the endpoints. Once this is understood, the general case follows rather easily.

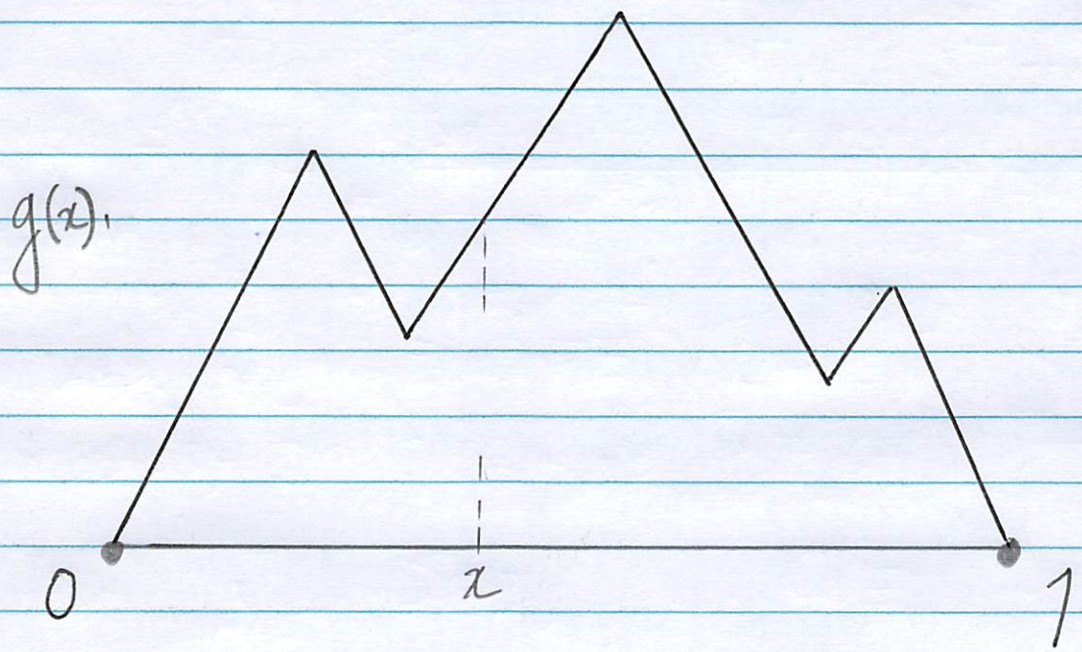
We shall focus on this case, then, and on a reward function

$$g : [0,1] \to [0,\infty)$$

which is continuous, with

$$g(0) = g(1) = 0.$$

Think Alpine landscapes, or TOBLERONE chocolate bars.



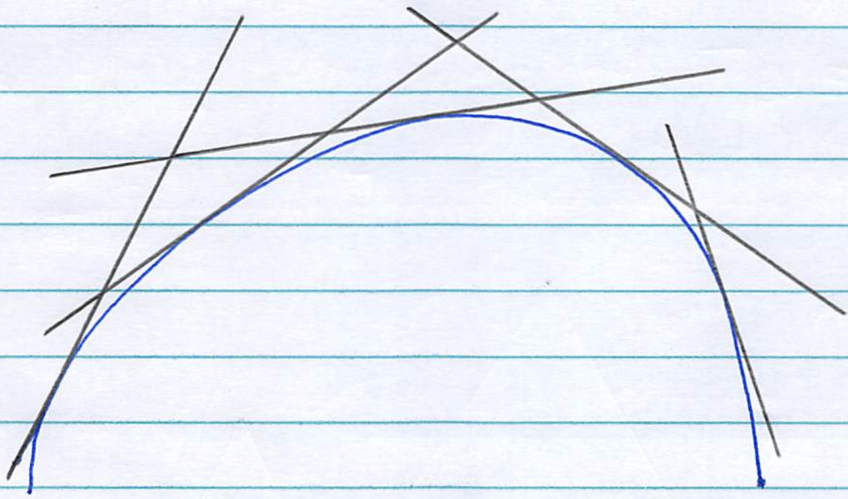
- Clearly $v(0) = v(1) = 0$, here.
- We also have: $v(\cdot) \le h(\cdot)$,

for every function $h:[0,1] \to [0,\infty)$ which dominates the reward $g(\cdot) \le h(\cdot)$ and is <u>superharmonic</u>:

$$h(x) \geq \mathbb{E}\left[h(W(\tau)) \,|\, W(0)=x \right], \quad \forall\, \tau \in \mathcal{S},\ x \in [0,1].$$

It is easy to check that this property holds for every $h:[0,1] \to [0,\infty)$ which is CONCAVE:

A concave function as the "pencil" of its tangents



And a little bit more effort, shows also the converse:

"<u>Every superharmonic</u> $h:[0,1] \to [0,\infty)$ <u>is necessarily concave.</u>"

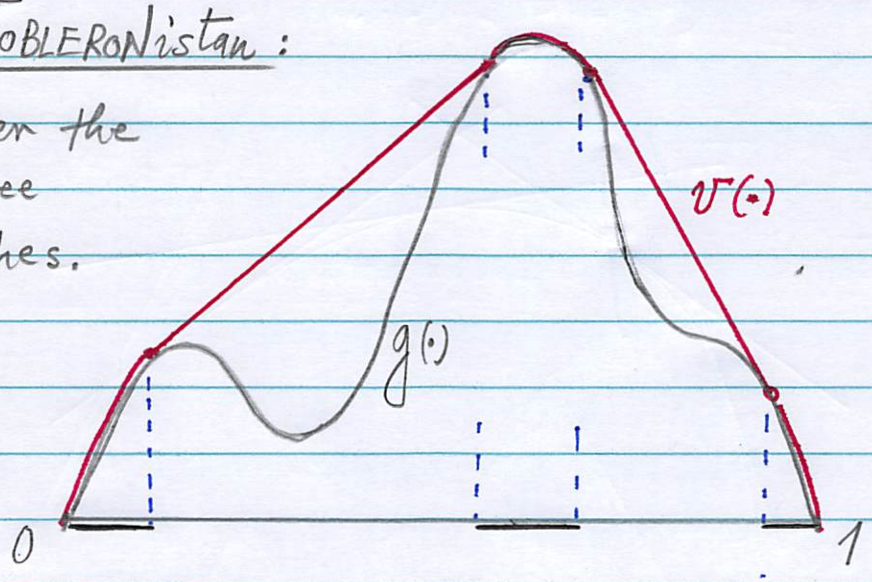
(A bit of Martingale Theory needed here...)

Then, the next and final step is to show that the maximal expected reward function $v(\cdot)$ from page 23 is <u>the smallest concave majorant of the reward function</u>:

$$v(x) = \min\{h(x) : h(\cdot) \text{ concave}, \ h(\cdot) \geq g(\cdot)\}$$
$$= \min\{\alpha + \beta x : \alpha + \beta y \geq g(y), \ \forall\, 0 \leq y \leq 1\}.$$

<u>Sunset over TOBLERONistan</u>:

Throw a veil over the landscape, see where it touches.



$\Sigma \triangleq \{x \in [0,1] : v(x) = g(x)\}$ Optimal Stop. Region

$\mathcal{O} \triangleq [0,1] \setminus \Sigma$ Optimal Continuation Region

$\hat{\tau}_* = \min\{t \geq 0 : W(t) \in \Sigma\}$ Optimal Stopping Time

Please note, that we have reduced a problem of <ins>Probability</ins> (optimally stopping a Brownian motion, or Random Walk) to a problem of <ins>Geometry</ins> (throwing a veil over the landscape delineated by the reward function).

But this problem has yet a third incarnation, this time in <ins>Analysis</ins>. Its optimal reward function v satisfies

$$-v'' \geq 0 \qquad \text{(Concavity)}$$

$$v - g \geq 0 \qquad \text{(Domination)}$$

$$(-v'')(v-g) = 0. \qquad \text{(Stop, or else Continue).}$$

For a given reward function g, finding a function v that satisfies these conditions is the <ins>Obstacle Problem</ins> of Analysis; and the boundary Γ separating the optimal continuation $\mathcal{O} = \{v > g\}$ from the optimal stopping $\Sigma = \{v = g\}$ region, is called the

<ins>Free-Boundary</ins>.

FINANCE

It turns out that in 1900, five years before EINSTEIN's great paper developing a physical theory for the Brownian movement, a Ph.D. student of the great Henri POINCARÉ named

Louis BACHELIER

wrote and "defended" a dissertation in which he also developed a similar theory – but motivated by the fluctuations of financial asset prices, not by the erratic movement of particles.

Both BACHELIER and his work were forgotten for nearly 75 years.

That is until, in the late 70's and early 80's, the realization dawned upon several folks that

- DOOB's theory of Martingales, and
- ITÔ's stochastic integral and calculus,

were just TAILOR-MADE for the needs of developing a mathematical theory for markets, portfolios, and the like.

For instance, the approximand

$$\sum_{j=1}^{2^n} H\left(\frac{(j-1)T}{2^n}\right)\left[W\left(\frac{jT}{2^n}\right)-W\left(\frac{(j-1)T}{2^n}\right)\right]$$

\# of shares of asset $W(\cdot)$ bought at the start of the "day"

price at the end of the "day"

price at the start of the "day"

for the ITÔ integral on p. 14, has a very clear interpretation in terms of total "profits and losses" over a finite number of "days".

The constraint, that the decision has to be made at the start of the day, reflects exactly the property that "the increments of the integrator stick out into the future".

There is an awful lot more about the connection with finance, of course.

We'll have to leave this for another day.

For more information and detail on these, and many more, things, please consult the excellent book of my good friend

BERNT ØKSENDAL : Stochastic Differential Equations. Universitext, Springer-Verlag.

You will need a solid foundation in the Theory of Probability. For this, consider the book of one of the greats:

DAVID WILLIAMS : Probability with Martingales. Cambridge University Press.